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EVOLUTION OF A DISCONTINUITY OF A VORTEX SHEET*

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The plane problem of the motion of a vortex sheet, which at the initial instant of time represents two sides of the angle $2\pi - \pi/k$ $(^1/_2 \leqslant k < 1)$, is considered. Motions of this kind are difficult to analyze because of the instability of the vortex sheet, regarded as a surface of tangential discontinuity, or, more accurately, because of the ill-posed nature of the Cauchy problem for the evolution equations of these surfaces /1, 2/. In problem of the flow around bodies, the evolution of the boundary of a jet, the flow of gas through permeable boundaries, the flow from a groove in a screen etc., double-spiral vortex formations are produced with an infinite number of spirals. This paper is devoted to a numerical calculation of these motions. In this case, the singularity is situated on the line of discontinuity itself, and not at its end. A simple problem of this kind was formulated in /3/, and involved determining the evolution of a wedge-shaped line of discontinuity. However, the method proposed for solving this problem in /3/, which reduces to a difference approximation of the initial differential equations, is unsatisfactory. Below we use the method of boundary integro-differential equations /4/, by means of which the dimensions of the problem can be reduced by one, and plane selfsimilar problems can be reduced to determining a function which depends on one variable.

Consider continuous selfsimilar flow round a corner $2\pi - \pi/k$, where $^1/_2 \leqslant k < 1$. The complex flow velocity inside the corner is zero, and outside it is given by

$$4nkat^{n(k-1)}z^{k-1}\exp[-i\pi(k-{}^1/_2)] \quad (z = x+iy) \tag{1}$$
$$z_1 = x_1 + iy_1 = at^n z$$

where n is the selfsimilarity index.

At the instant of time $t=0$ we replace the solid rays which bound the corner, by a line tangential to the discontinuity of velocity. Since the condition of equality of the effective pressure on the surface of the wedge when $t<0$ is not satisfied, the glide line will be deformed. Its form, together with the "initial" wedge, is shown in Fig.1.

The configuration of the surface separation, proposed in /3/, contains two inaccuracies. Firstly, the non-analytical nature of the solution, which occurs when $t \leqslant 0$ at the point $z=0$, must be preserved during the evolution. Consequently, the vortex point with zero circulation, situated at $t=0$ the vertex of the wedge, will be the centre of a double-spiral line of separation, as shown in Fig.1. This fact, ignored in /3/, was confirmed by numerical calculations. Secondly, the line of separation must not be symmetrical about the $y=0$ axis, and is wavy, since an external stimulus is necessary to set up the wave motions. This fact was also confirmed by numerical calculations.

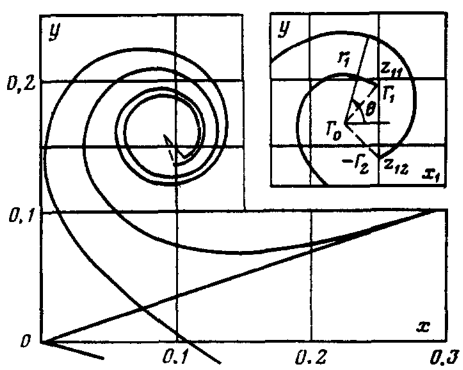

Fig.1

According to dimensional analysis, the flow is selfsimilar if $kn = 2n - 1$. In this case there is one independent dimensional constant a, which does not contain the dimensions of mass. We will require that at large distances from the vertex of the wedge the complex velocity asymptotically approaches the stationary value (1). In this case, at an infinitely distant point, the line of separation is wedge-shaped, as at $t=0$, and the pressure is constant. Then the condition for decay at infinity will have the form

$$z \to -(\pm G/4)^{1/k}\exp[\mp i\pi/(2k)] \text{ as } G \to \pm\infty \quad (\Gamma = na^2t^{2n-1}G) \tag{2}$$

where Γ is the dimensional circulation, measured from the centre of the spiral.

The flow considered can be obtained in the neighbourhood of any suddenly formed discontinuity on the gliding line. This will be the form of the

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flow in the near wake behind a short wing, having wedge-shaped side edges, both in the subsonic and supersonic flow of a gas for fairly small angles of attack, when the flow round the wing can be regarded as unbroken. Since the flow in considered in a small neighbourhood of the line of discontinuty, the assumption of selfsimilarity is justified.

Numerical calculations were carried out for the case when the line of discontinuity is a vortex sheet. The equation of the evolution of the vortex sheet $z = z(G)$ has the form

$$\bar{z}(G) - kG\frac{d\bar{z}(G)}{dG} = \frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{dG^\circ}{z(G) - z(G^\circ)} \tag{3}$$

The integration was carried out over the whole vortex sheet, and the particular integral represents the principal value of a Cauchy-type integral. The problem depends on one dimensionless parameter k, which represents the flare angle of the sheet at infinity.

If we know the solution $z = z(G)$ of the integro-differential equation (3) which satisfies condition (2), the field of flow outside the vortex sheet is determined by the dimensionless complex velocity

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{dG}{z - z(G)}$$

The integral here is not a particular integral since $z \neq z(G)$.

The infinite number of turns of the spiral discontinuity, characteristic for selfsimilar flows, is not amenable to numerical calculation, since the kernel of a double-spiral vortex sheet was approximated by a "discrete vortex-two section" scheme by analogy with the "discrete vortex-section" scheme used when investigating the evolution of single-spiral discontinuities /4/. The kernel of the double-spiral vortex sheet with an infinite number of turns $z_1 = z_1(\Gamma, t)$, corresponding to a change in the parameter Γ from $-\Gamma_2$ to Γ_1, was replaced by a point vortex with circulation $\Gamma_0 = \Gamma_1 + \Gamma_2$.

The two sections, connecting the end points of the spiral with the discrete vortex, divide the plane of the flow into two regions (the upper right part in Fig.1). The velocity of these sections is continuous, and the potential of the flow φ suffers a discontinuity. For the potentials to be unique in each of the regions of their jump $\Delta\varphi$, the following equations must be satisfied:

$$\Delta\varphi_1 = -\alpha\Gamma_0, \ \Delta\varphi_2 = -(1-\alpha)\Gamma_0$$

where α is an arbitrary parameter in this scheme, which can naturally be chosen so that the discontinuities of the potential are proportional to Γ_1 and Γ_2, i.e. $\alpha = \Gamma_1/\Gamma_2$, $\Delta\varphi_1 = -\Gamma_1$, $\Delta\varphi_2 = -\Gamma_2$.

We will require that the total force acting on the point vortex and both sections should be zero. We obtain

$$(z_{11} - z_{10})\frac{d\Gamma_1}{dt} + (z_{12} - z_{10})\frac{d\Gamma_2}{dt} + \left(\Phi_0 - \frac{dz_{10}}{dt}\right)\Gamma_0 = 0$$

$$z_{11} = z_1(\Gamma_1, t), \quad z_{12} = z_1(-\Gamma_2, t)$$

where z_{10} is the complex coordinate of the point vortex, and Φ_0 is the complex velocity at the point z_{10}.

The condition has the following form is selfsimilar variables:

$$\bar{z}_0 = [G_1\bar{z}(G_1) - G_2\bar{z}(G_2) - G_0\bar{z}_0]\,k/G_0 + \frac{1}{2\pi i}\int\limits_{-\infty}^{-G_2}\frac{dG}{z_0 - z(G)} + \frac{1}{2\pi i}\int\limits_{G_1}^{\infty}\frac{dG}{z_0 - z(G)} \tag{4}$$

When the dimensionless circulation of the point vortex G_0 is reduced, the number of turns in the spiral increases. Hence, the value of G_0 determines the error of this numerical scheme in the neighbourhood of the kernel.

To approximate the integrals in (3) and condition (4) for large values of $|G|$ we assumed that the sheets have the form (2) when $|G| > G_3$, where $G_3 > 0$ is fairly large. Then the integrals over these parts of the sheet can be expressed in the form of a hypergeometric series. The value of G_3 was found from the condition that the deviation of the shape of the sheet from the asymptotic form, represented by expression (2), was less than a specified value ε_0 for all $|G| > G_3$.

Hence, the value of G_3 determines the error of the approximation of the form of the sheet when $|G| \gg 1$.

In the intermediate regions, where $G_1 < G < G_3$ and $-G_3 < G < -G_2$ the vortex sheets can be represented by a series of discrete vortices. The form of the parts of the sheet defined by the coordinates of the discrete vortices, were found by an iterational method. We used form (2) as the initial approximation, while the coordinate z_0 was found from condition (4).

The iterations were carried out using (3) and condition (4) and was stopped when the discrepancy

$$\varepsilon = \max_{1 \leqslant j \leqslant N} |z_j^m - z_j^{m-1}|$$

became less than the specified amount ε_0 ($\simeq 10^{-4}$), where N is the total number of vortices, the subscript denotes the number of the discrete vortex in the sheet, and the superscript represents the number of the iteration.

In the next stage an additional vortex was added to each end of the vortex sheet, and the circulation of the neutral vortex was reduced bv an amount equal to the total circulation of the additional vortices. The iterational process was then begun again.

The build-up of the vortex sheet, and consequently, the further refinement of its form, was continued until the distance between the vortices decreased to ε_0 . The dependence of the change in the discrepancy on the number of iterations has a saw-tooth form with a decreasing amplitude. For a constant number of vortices, ε decreases rapidly and becomes less than ε_0. After building up the vortex sheet, ε increases abruptly, while the jumps decrease monotonically as N increases.

Numerical calculations were carried out for different values of k, and showed that double-spiral structures are feasible over the whole range $1/2 \leqslant k < 1$. Unlike /5/, a numerical solution could not be obtained for $k = 1$, as was also found in /6/. The vortex sheets obtained for $k =$ 0.56 and 0.71 are shown in Figs.1 and 2 respectively.

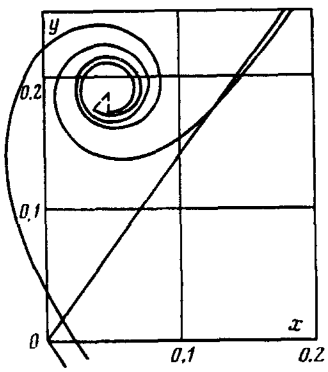

Fig.2

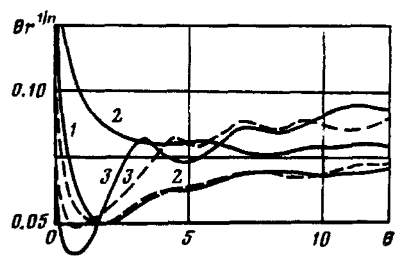

Fig.3

It is interesting to note that even when there are two-three turns in the spiral the numerical solution agrees satisfactorily with the asymptotic solution /7/. In polar self-similar coordinates r, θ with centre at the focus of the spiral (Fig.1) as $\theta \to -\infty$ we have $\theta = \theta_0 - a_0 r^{-1/n} + o\,(r^{-1/n})$. In Fig.3 we show that the results of numerical calculations differ only slightly from this relationship (curves *1*, *2* and *3* correspond to values of k of 0.5, 0.62, and 0.83; the continuous curves are for $G > 0$, and the dashed curves are for $G < 0$).

Obviously, the selfsimilar jet at the initial instant of time, which was formed in the experiments carried out by Yakimov at the edge of a prismatic cylinder incident on water /8/, is terminated by a double-spiral loop. This feature is an inherent property of particles of liquid which have acquired an "infinite" velocity at the instant of contact $t = 0$ with an edge of the incident body, and is possible in principle /9/. Unlike the algebraic form of a spiral vortex sheet (when $n > 1/2$) the free boundary is a logarithmic spiral, and converges rapidly to a constant scale. Hence, in experiments even for large Reynolds numbers, the turns of the spiral of a free boundary are difficult to detect.

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THE FIRST FUNDAMENTAL PROBLEM OF THE THEORY OF ELASTICITY FOR A SYMMETRIC LUNE*

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The first fundamental problem of the theory of elasticity is considered for a symmetric lune, when a symmetrically distributed normal load is specified on its boundary, and there are no tangential stresses. The problem is formulated and solved without preliminary reduction to the basic biharmonic problem. The proposed version and solution are based on the combined method of Fourier integrals and analysis of the Carleman problem /1, 2/. The problem of the stress state in a circular lune acted upon along the segments of its side surface by a uniform, normal compressive force was considered earlier in /3/, where the first fundamental problem of the theory of elasticity for a lune was reduced to the corresponding biharmonic problem.

1. The problem is formulated as follows /3/: to find the solution of the boundary value problem

$$\left[\frac{\partial^4}{\partial\alpha^4} + 2\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4}{\partial\beta^4} - 2\frac{\partial^2}{\partial\alpha^2} + 2\frac{\partial^2}{\partial\beta^2} + 1\right]\frac{\Phi}{h} = 0 \tag{1.1}$$

$$-\infty < \alpha < \infty, \quad -\gamma < \beta < \gamma$$

$$\left[\frac{1}{h}\frac{\partial^2}{\partial\alpha^2} - \operatorname{sh}\alpha\frac{\partial}{\partial\alpha} + \sin\beta\frac{\partial}{\partial\beta} - \cos\beta\right]\frac{\Phi}{h} = -\frac{q(\alpha)}{2} \tag{1.2}$$

$$-\infty < \alpha < \infty, \quad \beta = \pm\gamma$$

$$\frac{\partial^2}{\partial\alpha\,\partial\beta}\left[\frac{\Phi}{h}\right] = 0, \quad \beta = \pm\gamma; \quad h = \frac{1}{\operatorname{ch}\alpha + \cos\beta}$$

where $g(\alpha)/2$ is a given function characterizing the distributed load, and $\Phi(\alpha, \beta)$ is an unknown function. The symmetry of the stress state makes it possible to utilize the boundary conditions on the coordinate line $\beta = \gamma$ only, and we need consider only half of the region occupied by the lune $-\infty < \alpha < \infty$, $0 \leqslant \beta \leqslant \gamma$.

Applying the integral Fourier transformation to (1.1) and boundary conditions (1.2), we obtain

$$\frac{d^4W}{d\beta^4} + 2(1 - x^2)\frac{d^2W}{dx^2} + (x^2 + 1)^2 W = 0 \tag{1.3}$$

$$(x + i)^2 W(x + i, \gamma) + (x - i)^2 W(x - i, \gamma) + 2\cos\gamma x^2 W(x, \gamma) + i(x + i)W(x + i, \gamma) - i(x - i)W(x - i, \gamma) = G(x) \tag{1.4}$$

$$\left.\frac{dW}{d\beta}\right|_{\beta=\gamma} = 0$$

$$\left(W = V\left(\frac{\Phi}{h}\right), \ G = V(2q), \ V(f) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\alpha)\,e^{i\alpha x}\,d\alpha\right)$$

Let us write the general solution of (1.3) symmetrical with respect to β

$$W(x, \beta) = A(x)\operatorname{ch} x\beta\cos\beta + B(x)\operatorname{sh} x\beta\sin\beta \tag{1.5}$$

Substituting this solution into the second boundary condition of (1.4), we obtain a relation connecting $B(x)$ with $A(x)$, and from (1.5) we obtain

$$W(x, \beta) = A(x)\left[\operatorname{ch} x\beta\cos\beta + C(x)\operatorname{sh} x\beta\sin\beta\right] \tag{1.6}$$

$$C(x) = (\operatorname{tg}\gamma - x\operatorname{th} x\gamma)(\operatorname{tg}\gamma - \operatorname{th} x\gamma)^{-1}$$

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